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1998 J. Phys. A: Math. Gen. 31 2541

(http://iopscience.iop.org/0305-4470/31/11/006)

J. Phys. A: Math. Gen. **31** (1998) 2541–2549. Printed in the UK PII: S0305-4470(98)87254-5

Complex Langevin for semisimple compact connected Lie groups and *U*(1)

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Received 1 September 1997

Abstract. Several problems in quantum field theory like QCD at high densities lead to complex-valued actions S with $S : G \to \mathbb{C}$, where G is some group. Under the assumption that the complex Langevin process converges to a weakly stationary process we discuss the conditions under which it correctly simulates expectation values defined by complex weights. For technical reasons the discussion is restricted to $U(1)$ and semisimple compact connected Lie groups like $SU(n)$.

1. Introduction

There are several problems in physics which are defined by complex actions. Quantization is thus defined by integrals over complex-valued weights and standard numerical methods which rely on real positive weights are not applicable in this case. The complex Langevin (CL) has turned out to be quite useful in the calculation of integrals over complex-valued weight functions of the form e^{-2S}. S can be an action or also the Hamiltonian of some physical system. As pointed out in [1, 2] there is formally no restriction to a real-valued drift term for Langevin equations and thus a straightforward application of the CL is quite simple. Assuming a diffusion on $\mathbb{R}$ one has

$$dX = F(X)\,dt + dW \tag{1}$$

with the drift term

$$F(x) = -\frac{dS(x)}{dx}. \tag{2}$$

The formal continuation leads to

$$dZ = F(Z)\,dt + dW \tag{3}$$

where $dZ = dX + i\,dY$ and W_t is a real-valued Wiener process. By continuing to complex actions one unfortunately introduces two problems of uncertainty. In contrast to the process defined by a real action $S : G \to \mathbb{R}$, where roughly speaking the existence of a unique invariant measure can be proved (see, e.g., [19]), the extension to complex actions leads to singular diffusions (see equation (3)) and in general the existence of a unique invariant measure cannot be assumed. Events which might be connected with that problem have been observed numerically. The second problem is that, although the process has converged in some sense, the process will not necessarily give the right answer. This means that

0305-4470/98/112541+09$19.50

the stationary expectation value of some observable is not given by the integral over the complex-valued weight function. In this case the following equation is violated:

$$\lim_{t\to\infty} E(f(Z_t)) = \frac{1}{\mathcal{N}} \int_G f(x)\mathrm{e}^{-2S(x)}\, \mathrm{d}\mu(x) \tag{4}$$

where $Z_t \in \tilde{G}$. $\tilde{G}$ is some appropriate analytic continuation of the manifold G. Indeed the CL is known to sometimes give the wrong answer (see, e.g., [3]). There have been several approaches to understand the CL which mainly deal with processes on $\mathbb{R}^n$ vice versa it's analytic extension $\mathbb{C}^n$ [3–7]. In numerical simulations, however, it turned out that CL processes defined by actions on $U(1)$ in particular perform surprisingly well [8, 9]. Unfortunately there are only a few results for CLs on group spaces and these results mostly restrict to the definition of the Langevin equation or give hints for the numerical solution [10–13].

Since the singular structure of the time evolution operator (e.g., the Fokker–Planck operator) does not allow a general proof of the existence of a unique invariant measure for the CL process, one restricts the investigation to the case where it is assumed that one has certain stationary expectation values. In order to prove that these expectation values have the right properties (i.e. equation (4) holds), these expectation values have to fulfill certain conditions.

2. Orthogonal expansion of functions over groups

Before we continue with the CL we need to summarize a few results on unitary representations of Lie groups.

A unitary representation of a topological group G is a strongly continuous homomorphism $U : g \to U(g)$ from G into the group of unitary operators on some Hilbert space H called the representation space. Two unitary representations U, V with representation spaces H, H' are called unitarily equivalent if there exists a unitary isomorphism $T : H \to H'$ such that $T \circ U_g = V_g \circ T$ for every $g \in G$. We will denote the set of equivalence classes of equivalent irreducible representations by $\hat{G}$ and we write U^ρ for a member of the equivalence class ρ.

Let U be a unitary representation of some group G with representation space H. An element $x_0 \in H$ is called a cyclic vector if the linear span of the set $\{U_g x_0 \mid g \in G\}$ is dense in H. If U has at least one cyclic vector then the representation U is called cyclic.

It can be proved [14–16] that any unitary representation is the direct sum of cyclic representations and that a finite dimensional unitary representation U of a group G can be decomposed into a direct sum of irreducible representations. In particular one has that any unitary representation U of a compact group G is a direct sum of finite-dimensional irreducible unitary representations and that any irreducible unitary representation of a compact group is finite-dimensional.

Let $\rho \in \hat{G}$ and let U^ρ be in ρ. Choose a fixed but arbitrary basis $(\mathrm{e}_j^\rho)_{1\leqslant j\leqslant n}$ in the representation space H^ρ with $\mathbb{C}$-dimension n_ρ and define the following continuous functions on G:

$$u_{jk}^\rho : \begin{cases} G \to \mathbb{C} \\ s \mapsto (U_s^\rho e_k^\rho, e_j^\rho) \qquad 1 \leqslant j, k \leqslant n_\rho. \end{cases} \tag{5}$$

The functions u_{jk}^ρ are called coordinate functions. For these coordinate functions one has the following orthogonality relations [14–16]:

$$(u_{jk}^\rho, u_{hl}^{\rho'}) = 0 \qquad 1 \leqslant j, k \leqslant n_\rho \quad 1 \leqslant h, l \leqslant n_{\rho'} \tag{6}$$

if $\rho \neq \rho'$ and

$$(u^\rho_{jk}, u^\rho_{hl}) = \frac{1}{n_\rho}\delta_{jh}\delta_{kl} \qquad 1 \leqslant j,k \leqslant n_\rho \quad 1 \leqslant h,l \leqslant n_\rho. \tag{7}$$

The set of functions $n_\rho^{1/2}u^\rho_{jk}$ form an orthonormal basis for $L^2(G)$ with G a compact group. Therefore for $f \in L^2(G)$ we have

$$f = \sum_{\rho\in\hat{G}}\sum_{j,k=1}^{n_\rho} n_\rho a^\rho_{jk}u^\rho_{jk} \tag{8}$$

where $a^\rho_{jk} = (f, u^\rho_{jk}) = \int_G f\overline{u}^\rho_{jk}\,\mathrm{d}\mu$. This result is also known as the *Peter–Weyl* theorem [14–16].

For $\rho \in \hat{G}$ let us consider the continuous function

$$\chi^\rho : \begin{cases} G \to \mathbb{C} \\ g \mapsto \operatorname{Tr} U^\rho_g = \sum_{k=1}^{n_\rho} u^\rho_{kk}. \end{cases} \tag{9}$$

This function χ^ρ is called the character of ρ and makes sense, since it depends only on the equivalence class ρ. Moreover, the characters are *central* functions, i.e. functions obeying

$$f(g^{-1}hg) = f(h) \qquad \text{or equivalently} \qquad f(gh) = f(hg) \qquad \forall\, g \in G.$$

Indeed

$$\chi^\rho(g^{-1}hg) = \operatorname{Tr} U^\rho_{g^{-1}hg} = \operatorname{Tr}(U^\rho_{g^{-1}}U^\rho_h U^\rho_g) = \operatorname{Tr} U^\rho_h = \chi^\rho(h).$$

For χ^ρ we further have [15, 16]

(i) If U^ρ and $U^{\rho'}$ are equivalent then $\chi^\rho = \chi^{\rho'}$;
(ii) $\chi^\rho(ghg^{-1}) = \chi^\rho(h)$;
(iii) $\chi^{\rho\oplus\rho'} = \chi^\rho + \chi^{\rho'}$;
(iv) $\chi^{\rho\otimes\rho'} = \chi^\rho\chi^{\rho'}$;
(v) $\overline{\chi^\rho(g)} = \chi^\rho(g^{-1})$;
(vi) $\chi^\rho(e) = \dim_{\mathbb{C}} H_\rho$.

For these characters we also have orthogonality relations [14–16]

$$(\chi^\rho, \chi^{\rho'}) = 0 \qquad \rho \neq \rho' \tag{10}$$

and

$$(\chi^\rho, \chi^{\rho'}) = 1 \qquad \rho = \rho'. \tag{11}$$

Whereas the functions $(n_\rho^{1/2}u^\rho_{jk})$ build an orthonormal basis of $L^2(G)$, the characters $(\chi^\rho)_{\rho\in\hat{G}}$ do so for the central functions. For every central function $f \in L^2(G)$ we have

$$f = \sum_{\rho\in\hat{G}}(f, \chi^\rho)\chi^\rho. \tag{12}$$

Let $\mathcal{T}_\rho(G)$ denote the linear space spanned by the coordinate functions u^ρ_{jk}. If G is now a real compact connected semisimple Lie group and $\Omega(\mathfrak{g})$ the Casimir element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ then the functions of $\mathcal{T}_\rho(G)$ are eigenfunctions of $\Omega(\mathfrak{g})$ considering the Casimir element as a differential operator [14].

3. The complex Langevin

Let M be some n-dimensional manifold. Let $W^i, 1 \leqslant i \leqslant n$ real Wiener processes and assume $n+1$ vector fields $\mathcal{A}_i, 0 \leqslant i \leqslant n$. Then with $f \in C_0^\infty(M)$ (for a compact manifold it is sufficient to use $f \in C^\infty(M)$) we define a stochastic differential equation (SDE) [19, 20] by

$$\mathrm{d}(f \circ X) = (\mathcal{A}_0 f)(X)\,\mathrm{d}t + \sum_{i=1}^{n} (\mathcal{A}_i f)(X) * \mathrm{d}W^i \tag{13}$$

where $\sum_{i=1}^{n} (\mathcal{A}_i f)(X) * \mathrm{d}W^i$ is the stochastic differential in the sense of Stratonovich denoted by $*$.

It can be proved [19, 20] that for such an SDE there exists a unique solution $(X_t)_{t<\zeta}$ with lifetime $\zeta > 0$ starting at $X_0 = x_0$.

Consider the above situation: for $t < \zeta$ we then have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(f \circ X_t) = E((Lf) \circ X_t) \tag{14}$$

with

$$L = \mathcal{A}_0 + \frac{1}{2} \sum_{i=1}^{n} \mathcal{A}_i^2. \tag{15}$$

To prove this one has to apply among other results the important relation between Stratonovich and Ito differentials, which is $X * \mathrm{d}Y = X\,\mathrm{d}Y + \frac{1}{2}\mathrm{d}X\,\mathrm{d}Y$. For details see, e.g., [19, 20].

The adjoint operator L^* is often called the Fokker–Planck operator, especially in the case where $M = \mathbb{R}^n$. Under certain conditions it can be proved that the operator L^* defines a unique invariant measure with

$$\lim_{t\to\infty} E(f \circ X_t) = \int_M f(x)\nu(x)\,\mathrm{d}\mu(x) \tag{16}$$

where

$$L^*\nu = 0 \tag{17}$$

and $\mathrm{d}\mu$ could be the Haar measure on a group or any other appropriate volume measure [19, 20].

Let G be a real analytic Lie group denoted from now on as real Lie group. A complex analytic Lie group just denoted as a complex Lie group $G_\mathbb{C}$, together with a homomorphism $\gamma : G \to G_\mathbb{C}$, is called the *universal complexification* of G if every homomorphism $\alpha : G \to H$ into a complex Lie group H can be extended to a holomorphic homomorphism $\alpha^\mathbb{C} : G_\mathbb{C} \to H$ such that $\alpha^\mathbb{C} \circ \gamma = \alpha$ [17, 16].

For such a universal complexification one has the following results [17, 16]:

(i) The universal complexification exists for any real connected Lie group.
(ii) In the case of a compact group G the homomorphism $\gamma : G \to G_\mathbb{C}$ is injective and $\gamma(G)$ is a compact subgroup of $G_\mathbb{C}$.
(iii) If $L(G)$ denotes the Lie algebra of G then

$$\mathbb{C} \otimes_\mathbb{R} L(G) =: L(G)_\mathbb{C} \simeq L(G_\mathbb{C}).$$

(iv) If G is semisimple then so is $G_\mathbb{C}$.

For the groups $SU(n)$ and $U(n)$ which are important in physics we have the respective relations

$$SU(n)_{\mathbb{C}} \simeq SL(n, \mathbb{C})$$

$$U(n)_{\mathbb{C}} \simeq GL(n, \mathbb{C}).$$

Let G be a compact group and note that $u : G \to U(n),\ g \mapsto u^{\rho}_{ij}(g)_{1\leqslant i,j\leqslant n}$ is a unitary matrix representation in $\mathbb{C}^n,\ n = \dim \rho$. The existence of a complexification implies that u can be extended to some holomorphic matrix representation $u^{\mathbb{C}}$, or in other words, the coordinate functions u^{ρ}_{ij} have holomorphic extensions $u^{\rho,\mathbb{C}}_{ij}$.

From now on we think of G as a subgroup of $G_{\mathbb{C}}$. A complex Lie group of dimension n may be regarded as a real Lie group of dimension $2n$. Moreover, in the case of a real connected Lie group remark (iii) shows that if $(A_a)_{1\leqslant a\leqslant n}$ is an $\mathbb{R}$-basis of $L(G)$ then $(A_a)_{1\leqslant a\leqslant n}$ is a $\mathbb{C}$-basis of $L(G_{\mathbb{C}})$. Thus $((A_a)_{1\leqslant a\leqslant n}, (iA_a)_{1\leqslant a\leqslant n})$ is an $\mathbb{R}$-basis of $L(G_{\mathbb{C}})$ when considered as a real Lie algebra.

Consequently, if we regard $G_{\mathbb{C}}$ either as a complex or real manifold we may distinguish the following derivations. Take a basis element $A_a \in L(G_{\mathbb{C}})$ and $f \in \mathcal{O}(G_{\mathbb{C}})$ (we denote the set of holomorphic functions by $\mathcal{O}(G_{\mathbb{C}})$) then on the complex Lie group we have

$$(\mathcal{A}_a f)(s) = \frac{\mathrm{d}}{\mathrm{d}z} f(s \exp z A_a)\bigg|_{z=0} \qquad s \in G_{\mathbb{C}} \tag{18}$$

where $\Phi : \mathbb{C} \times G_{\mathbb{C}} \to G_{\mathbb{C}},\ (z, s) \mapsto s \exp z A_a$ is the analytic flow of the left invariant vector field $\mathcal{A}_a$ corresponding to A_a.

In the real case we distinguish

$$(\mathcal{A}^x_a f)(s) = \frac{\mathrm{d}}{\mathrm{d}t} f(s \exp t A_a)\bigg|_{t=0} \qquad s \in G_{\mathbb{C}} \quad f \in C^{\infty}(G_{\mathbb{C}}) \tag{19}$$

and

$$(\mathcal{A}^y_a f)(s) = \frac{\mathrm{d}}{\mathrm{d}t} f(s \exp t \mathrm{i} A_a)\bigg|_{t=0} \qquad s \in G_{\mathbb{C}} \quad f \in C^{\infty}(G_{\mathbb{C}}) \tag{20}$$

where $(t, s) \mapsto (s \exp t A_a)$ $\big((t, s) \mapsto (s \exp t \mathrm{i} A_a)\big)$ are the two restrictions to $\mathbb{R} \times G_{\mathbb{C}}$ of the flows corresponding to the two basis elements A_a ($\mathrm{i}A_a$). Note that in (19) and (20) the superscripts x and y have only notational meaning.

In the case where $f \in \mathcal{O}(G_{\mathbb{C}})$ we have

$$\mathcal{A}_a f = \mathcal{A}^x_a f = -\mathrm{i}\mathcal{A}^y_a f. \tag{21}$$

Finally, restricting Φ to $\mathbb{R} \times G \to G$ we obtain the derivatives on G:

$$(\mathcal{A}^{\mathrm{r}}_a f)(s) = \frac{\mathrm{d}}{\mathrm{d}t} f(s \exp t A_a)\bigg|_{t=0} \qquad s \in G \quad f \in C^{\infty}(G) \tag{22}$$

(the superscript 'r' stands for restriction to G).

We define the CL equation as the following stochastic differential equation on $G_{\mathbb{C}}$ regarded as a real manifold. With $S \in C^{\infty}(G_{\mathbb{C}})$ we demand that $\forall\, f \in C^{\infty}(G_{\mathbb{C}})$

$$\mathrm{d}(f \circ X) = -\sum_a (\theta_a(\mathcal{A}^x_a f))(X)\,\mathrm{d}t - \sum_a (\phi_a(\mathcal{A}^y_a f))(X)\,\mathrm{d}t + \sum_a (\mathcal{A}^x_a f)(X) * \mathrm{d}W^a \tag{23}$$

where

$$\theta_a = \tfrac{1}{2}(\mathcal{A}^x_a \mathrm{Re}\, S + \mathcal{A}^y_a \mathrm{Im}\, S)$$

$$\phi_a = \tfrac{1}{2}(\mathcal{A}^x_a \mathrm{Im}\, S - \mathcal{A}^y_a \mathrm{Re}\, S).$$

When $f, S \in \mathcal{O}(G_{\mathbb{C}})$, using (21) equation (23) simplifies to

$$\mathrm{d}(f \circ X) = -\sum_a (\mathcal{A}_a S)(\mathcal{A}_a f)(X)\,\mathrm{d}t + \sum_a (\mathcal{A}_a f)(X) * \mathrm{d}W^a. \tag{24}$$

For $f, S \in \mathcal{O}(G_{\mathbb{C}})$, assuming that $E(f \circ X_t)$ exists, it follows from equation (24) that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(f \circ X_t) = E((L^{\mathbb{C}} f) \circ X_t) \tag{25}$$

where $L^{\mathbb{C}} = \frac{1}{2}\sum_a \mathcal{A}_a^2 - \sum_a (\mathcal{A}_a S)\mathcal{A}_a$.

Let $\overline{G}$ be a complex analytic semisimple group, G a compact real form of $\overline{G}$, and N a connected open neighbourhood of G in $\overline{G}$. If f is holomorphic on N such that $f|_G = 0$, then $f = 0$ on N [18]. This result is also known as Weyl's unitarian trick. This is the generalization of the well known result for a holomorphic function of one variable.

With the aid of Weyl's unitarian trick we can show now that the functions $u_{ij}^{\rho,\mathbb{C}}$ are eigenfunctions of $\Omega^{\mathbb{C}} := \frac{1}{2}\sum_a \mathcal{A}_a^2$ whenever G is a real connected semisimple Lie group:

$$\Omega^{\mathbb{C}} u_{ij}^{\rho,\mathbb{C}}\big|_G = \Omega^{\mathrm{r}} u_{ij}^{\rho} = \lambda_\rho u_{ij}^{\rho} \tag{26}$$

$$\downarrow \text{ holomorphic extension}$$

$$\Omega^{\mathbb{C}} u_{ij}^{\rho,\mathbb{C}} \stackrel{!}{=} \lambda_\rho u_{ij}^{\rho,\mathbb{C}}. \tag{27}$$

In equation (27) $\stackrel{!}{=}$ follows from the unitarian trick, since the equality holds on G.

Next we list two equalities that will be needed in what follows.

(i) $\forall s \in G_{\mathbb{C}}$:

$$(\mathcal{A}_a u_{jk}^{\rho,\mathbb{C}})(s) = \frac{\mathrm{d}}{\mathrm{d}z}(u_{jk}^{\rho,\mathbb{C}}(s \exp z A_a))\bigg|_{z=0} \tag{28}$$

$$= \frac{\mathrm{d}}{\mathrm{d}z}\Big(\sum_h u_{jh}^{\rho,\mathbb{C}}(s) u_{hk}^{\rho,\mathbb{C}}(\exp z A_a)\Big)\bigg|_{z=0} \tag{29}$$

$$= \sum_h u_{jh}^{\rho,\mathbb{C}}(s) \frac{\mathrm{d}}{\mathrm{d}z}(u_{hk}^{\rho,\mathbb{C}}(\exp z A_a))\bigg|_{z=0} \tag{30}$$

$$= \sum_h u_{jh}^{\rho,\mathbb{C}}(s) B_{a,hk}^{\rho}. \tag{31}$$

(ii) $\forall s \in G$:

$$\mathcal{A}_a^{\mathrm{r}} u_{jk}^{\rho}(s) = \sum_h u_{jh}^{\rho}(s) B_{a,hk}^{\rho} \tag{32}$$

because for analytic functions we have $\mathrm{d}/\mathrm{d}z = \mathrm{d}/\mathrm{d}t$.

The matrices $(B_{a,hk}^{\rho})_{1\leqslant h,k\leqslant \dim\rho} \in M(\dim\rho, \mathbb{C})$ are the Lie algebra basis elements of the matrix representation $u^{\mathbb{C}}$ (u).

Now we consider a stochastic process X on the complexification $G_{\mathbb{C}}$ of a real connected compact semisimple Lie group G obeying the complex Langevin equation.

We define $\Lambda_{t,ij}^{\rho} := E(u_{ij}^{\rho,\mathbb{C}} \circ X_t)$ and let S have the expansion

$$S = \sum_{\rho\in\Theta}\sum_{l,m} c_{lm}^{\rho} u_{lm}^{\rho} \quad \Rightarrow \quad S^{\mathbb{C}} = \sum_{\rho\in\Theta}\sum_{l,m} c_{lm}^{\rho} u_{lm}^{\rho,\mathbb{C}}$$

with Θ a finite subset of $\hat{G}$. If we assume that

(i) the expectation values become stationary so that we may define $\Lambda^{\rho}_{ij} := \lim_{t\to\infty} E(u^{\rho,\mathbb{C}}_{ij} \circ X_t)$ and

$$\sum_{\rho\in\hat{G}} \sum_{i,j} n_\rho |\Lambda^{\rho}_{ij}|^2 < \infty \tag{33}$$

then there is a $h \in L^2(G)$ such that

$$\Lambda^{\rho}_{ij} = \int u^{\rho}_{ij} h \, \mathrm{d}\mu \tag{34}$$

where h is given by

$$h = \sum_{\rho\in\hat{G}} \sum_{i,j} n_\rho \Lambda^{\rho}_{ij} \overline{u^{\rho}_{ij}}. \tag{35}$$

(ii)

$$\lim_{t\to\infty} E((u^{\rho,\mathbb{C}}_{lp} u^{\hat{\rho},\mathbb{C}}_{iq}) \circ X_t) = \int u^{\rho}_{lp} u^{\hat{\rho}}_{iq} h \, \mathrm{d}\mu$$

$$\forall \rho, \hat{\rho} \in \hat{G} \quad \forall 1 \leqslant l, p \leqslant \dim\rho \quad 1 \leqslant i, q \leqslant \dim\hat{\rho} \tag{36}$$

then $Fh = 0$ in the sense of distributions with

$$F := \sum_a \mathcal{A}^{\mathrm{r}}_a \left(\tfrac{1}{2}\mathcal{A}^{\mathrm{r}}_a + (\mathcal{A}^{\mathrm{r}}_a S)\right). \tag{37}$$

Since F is elliptic, h is $\in C^\infty(G)$ (by Weyl's lemma). Note that 0 is an isolated eigenvalue of F and the corresponding normalized eigenvector is $h = (1/\mathcal{N})\mathrm{e}^{-2S}$ with $\mathcal{N} = \int \mathrm{e}^{-2S}\,\mathrm{d}\mu$, assuming $0 < \mathcal{N} < \infty$.

This can be seen as follows. $\forall \hat{\rho} \in \hat{G}$, $\forall 1 \leqslant i, j \leqslant n_\rho$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{\hat{\rho}}_{t,ij} = E\left(\left\{\frac{1}{2}\sum_a \mathcal{A}^2_a u^{\hat{\rho},\mathbb{C}}_{ij} - \sum_a (\mathcal{A}_a S^{\mathbb{C}})(\mathcal{A}_a u^{\hat{\rho},\mathbb{C}}_{ij})\right\} \circ X_t\right) \tag{38}$$

by equation (25)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{\hat{\rho}}_{t,ij} = E\left(\lambda_\rho u^{\hat{\rho},\mathbb{C}}_{ij} \circ X_t\right) - \sum_a \sum_{\rho\in\Theta} \sum_{l,m} \sum_p \sum_q c^{\rho}_{lm} B^{\rho}_{a,pm} B^{\hat{\rho}}_{a,qj} \left((u^{\rho,\mathbb{C}}_{lp} u^{\hat{\rho},\mathbb{C}}_{iq}) \circ X_t\right) \tag{39}$$

by equations (27), (31)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{\hat{\rho}}_{t,ij} = \lambda_\rho E\left(u^{\hat{\rho},\mathbb{C}}_{ij} \circ X_t\right) - \sum_{\text{all indices}} c^{\rho}_{lm} B^{\rho}_{a,pm} B^{\hat{\rho}}_{a,qj} E\left((u^{\rho,\mathbb{C}}_{lp} u^{\hat{\rho},\mathbb{C}}_{iq}) \circ X_t\right) \tag{40}$$

$$\lim_{t\to\infty}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{\hat{\rho}}_{t,ij}\right) = \lambda_\rho \int u^{\hat{\rho}}_{ij} h \, \mathrm{d}\mu - \sum_{\text{all indices}} c^{\rho}_{lm} B^{\rho}_{a,pm} B^{\hat{\rho}}_{a,qj} \int u^{\rho}_{lp} u^{\hat{\rho}}_{iq} h \, \mathrm{d}\mu \tag{41}$$

$$= \int (\Omega^{\mathrm{r}} u^{\hat{\rho}}_{ij}) h \, \mathrm{d}\mu - \sum_a \int (\mathcal{A}^{\mathrm{r}}_a S)(\mathcal{A}^{\mathrm{r}}_a u^{\hat{\rho}}_{ij}) h \, \mathrm{d}\mu = 0 \tag{42}$$

because of equations (26), (32) and stationarity.

Clearly, $h \in L^2(G)$ may be regarded as a regular distribution. To show

$$(Fh)(\phi) = 0 \qquad \forall \phi \in C^\infty_0(G) = C^\infty(G)$$

it is sufficient to show that

$$(Fh)(u^{\rho}_{ij}) = 0 \qquad \forall \rho \in \hat{G} \quad \forall 1 \leqslant i, j \leqslant n_\rho$$

since every $\phi \in C^\infty(G)$ has a harmonic expansion in terms of the u^ρ_{ij}. However, this is already expressed in equation (42).

The result above remains valid for arbitrary $S \in C^\infty(G)$. If S now has the harmonic expansion

$$S = \sum_{\rho\in\hat{G}} \sum_{l,m} c^\rho_{lm} u^\rho_{lm} \tag{43}$$

and

$$S^{\Theta_n} = \sum_{\rho\in\Theta_n} \sum_{i,j} c^\rho_{lm} u^\rho_{lm} \qquad n \geqslant 0 \tag{44}$$

is any sequence of truncated expansions with finite subsets $\Theta_n \subset \hat{G}$ then $\lim_{n\to\infty} \|S^{\Theta_n} - S\|_2 = 0$. Moreover, since $S \in C^\infty(G)$ the convergence is even uniform like it is for $\exp(-2S^{\Theta_n}) \to \mathrm{e}^{-2S}$. The calculations above show that $\forall\, \Theta_n$

$$\Lambda^{\rho,\Theta_n}_{ij} := \lim_{t\to\infty} \Lambda^{\rho,\Theta_n}_{t,ij} = \frac{1}{\mathcal{N}} \int u^\rho_{ij} \exp\bigl(-2S^{\Theta_n}\bigr)\, \mathrm{d}\mu \tag{45}$$

where $\Lambda^{\rho,\Theta_n}_{t,ij} := E(u^\rho_{ij} \circ X^{\Theta_n}_t)$ are the expectation values obtained from the process corresponding to the action S^{Θ_n}. Now by the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \Lambda^{\rho,\Theta_n}_{ij} = \frac{1}{\mathcal{N}} \int u^\rho_{ij} \mathrm{e}^{-2S}\, \mathrm{d}\mu. \tag{46}$$

3.1. The complex Langevin on U(1)

The Abelian multiplicative group $U(1)$ has the complexification $U(1)_\mathbb{C} = \mathbb{C}\backslash\{0\}$. $U(1)$ is isomorphic to the additive torus group $T := \mathbb{R}/2\pi\mathbb{Z}$ and we study the diffusion on T. A function on T may be considered as a function on $\mathbb{R}$ having period 2π and harmonic analysis on T is the well known Fourier analysis. Every $f \in L^2(T)$ has an expansion

$$f = \sum_q c_q \mathrm{e}^{\mathrm{i}qx} \qquad \text{with} \quad c_q = \int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}qx}\, \mathrm{d}\mu(x) \tag{47}$$

where $\mathrm{d}\mu(x) = \mathrm{d}x/2\pi$. The considerations above immediately carry over to the present case. The vector fields are now $\mathcal{A} = \mathrm{d}/\mathrm{d}z$, $\mathcal{A}^\mathrm{r} = \mathrm{d}/\mathrm{d}x$ and equation (25) has the translation

$$\frac{\mathrm{d}}{\mathrm{d}t} E(f \circ X_t) = E\left(\left\{\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} f\right) - \left(\frac{\mathrm{d}}{\mathrm{d}z} S\right)\left(\frac{\mathrm{d}}{\mathrm{d}z} f\right)\right\} \circ X_t\right). \tag{48}$$

Again suppose that S can be expressed as a finite series

$$S(x) = \sum_q c_q \mathrm{e}^{\mathrm{i}qx} \tag{49}$$

and define

$$\Lambda^k_t := E(\mathrm{e}^{\mathrm{i}kX_t}). \tag{50}$$

As before we assume that the expectation values become stationary with

$$\Lambda^k := \lim_{t\to\infty} E\left(\mathrm{e}^{\mathrm{i}kX_t}\right) \tag{51}$$

and

$$\sum_{q\in\mathbb{Z}} |\Lambda^q|^2 < \infty \tag{52}$$

then with $h := \sum_q \Lambda^q e^{-iqx}$ we have $\Lambda^k = \int e^{ikx} h \, d\mu$.

In the limit $t \to \infty$, and with equation (48), we obtain

$$0 = -\frac{k^2}{2}\Lambda^k + k\sum_q q c_q \Lambda^{q+k} \tag{53}$$

$$= -\frac{k^2}{2}\int e^{ikx} h \, d\mu + \int k \sum_q q c_q e^{i(q+k)x} h \, d\mu \tag{54}$$

$$= \frac{1}{2}\int \left(\frac{d^2}{dx^2} e^{ikx}\right) h \, d\mu - \int \left(\left(\frac{d}{dx} S\right) h\right) \left(\frac{d}{dx} e^{ikx}\right) \, d\mu. \tag{55}$$

The same arguments we used above shows that

$$Fh = 0 \qquad \text{(distributional)}$$

with

$$F = \frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx} + \left(\frac{d}{dx} S\right)\right)$$

so that the normalized solution is given by $h = (1/\mathcal{N})e^{-2S}$. As in the previous case following the same arguments it is also true that $\Lambda^k = \int e^{ikx} e^{-2S} \, d\mu$ for arbitrary $S \in C^\infty(T)$.

4. Conclusions

In this paper we have discussed the general environment of the CL (roughly speaking) on groups which have an important impact on physics. We have also identified certain conditions on such CL processes which now permit a verification of the correctness of the process on these manifolds. Unfortunately we are still unable to present a complete theory of CLs which would include criteria which allow proof of convergence at all.

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